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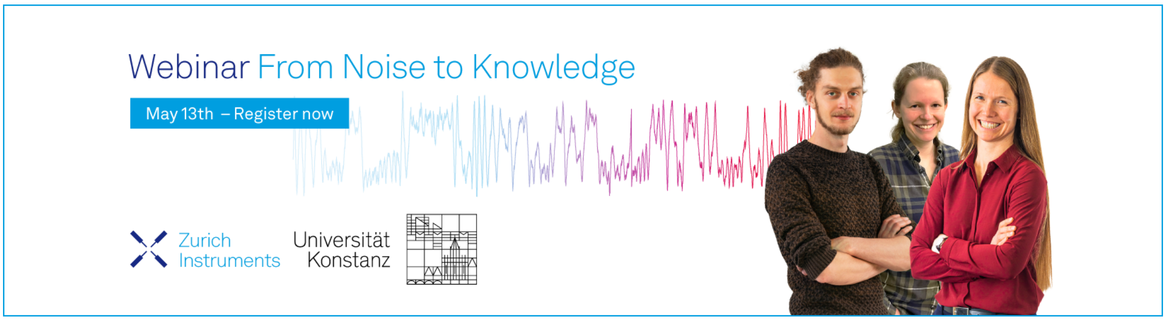

Subclasses of Univalent Functions Associated with Generalized q-Mittag-Leffler Function

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Abstract. In this present article, we establish new subclasses of analytic univalent functions defined by a differential operator associated with generalized q-Mittag-Leffler function. Further, we obtain coefficient theorems, extremal properties, radius of starlikeness, radius of convexity, radius of close-to-convexity, closure theorems for these classes.

INTRODUCTION

Let $\mathscr{A}$ be the class of analytic functions given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open unit disc $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = 0$ and $f'(0) = 1$. Let $\mathscr{S}$ be the subclass of $\mathscr{A}$ having functions of the form (1) which are univalent $\mathscr{U}$. Let $\mathbb{T}$ be the subclass of $\mathscr{S}$ consisting all functions from $\mathscr{S}$ those who have negative coefficients which is given by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad a_n > 0,\ z \in \mathscr{U}. \tag{2}$$

This class was earlier studied by Silverman [1].
Now, if $f(z)$ and $h(z)$ belong to the class $\mathscr{S}$ then the convolution of $f(z)$ and $h(z)$ is denoted by $f * h$ and is given by

$$f * h = h * f = z + \sum_{n=2}^{\infty} a_n b_n z^n$$

where,
$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.
The Mittag-Leffler function $\mathscr{H}_\sigma(z)$ was introduced by Mittag-Leffler [2] and it is given by

$$\mathscr{H}_\sigma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n + 1)}, \qquad \sigma, z \in \mathbb{C}, \Re(\sigma) > 0,$$

where Γ is Gamma function.
Wiman [3] introduced following generalized Mittag-Leffler function

$$\mathscr{H}_{\sigma,\delta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n + \delta)}, \qquad \sigma, \delta, z \in \mathbb{C}, \Re(\sigma) > 0, \Re(\delta) > 0.$$

Prabhakar introduced [4] the following function

$$\mathscr{H}^{\tau}_{\sigma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\tau)_n}{\Gamma(\sigma n + \delta)} \frac{z^n}{n!}, \qquad \sigma, \delta, \tau, z \in \mathbb{C}, \Re(\sigma) > 0, \Re(\delta) > 0, \Re(\tau) > 0.$$

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Later, Shukla and Prajapati [5] defined another generalized Mittag-Leffler function

$$\mathscr{H}_{\sigma,\delta}^{\tau}(z)=\sum_{n=0}^{\infty}\frac{(\tau)_{pn}}{\Gamma(\sigma n+\delta)}\frac{z^n}{n!},\qquad \sigma,\delta,\tau,z\in\mathbb{C},\Re(\sigma)>0,\Re(\delta)>0,\Re(\tau)>0.$$

where $p\in(0,1)\cup\mathbb{N}$ and $(\tau)_{pn}$ denotes the generalized Pochhammer symbol which is in particular reduces to $p^{pn}\prod_{r=1}^{p}\left(\frac{\tau+r-1}{p}\right)_n$ if $p\in\mathbb{N}$.

Definition 1. *[6] "Let $0<q<1$, then $[n]_q!$ denotes the q-factorial, which is defined as follows.*

$$[n]_q!=\begin{cases}[n]_q[n-1]_q\cdots[2]_q[1]_q, & n=1,2,3,\cdots\\ 1, & n=0\end{cases}$$

where, $[n]_q=\frac{1-q^n}{1-q}=1+\sum_{m=1}^{n-1}q^m$ and $[0]_q=0$."

Definition 2. *[6] "The q-generalized Pochhammer symbol $[\upsilon]_{qn},\upsilon\in\mathbb{C}$, is given as*

$$[\upsilon]_{qn}=[\upsilon]_q[\upsilon+1]_q[\upsilon+2]_q\cdots[\upsilon+n-1]_q$$

and the q-gamma function is defined as $\Gamma_q(\upsilon+1)=[\upsilon]_q\Gamma_q(\upsilon)$ and $\Gamma_q(1)=1$. It follows that $\Gamma_q(n+1)=[n]_q!$."

In [7] presented a generalized q-Mittag-Leffler function as follows

$$\mathscr{H}_{\sigma,\delta}^{\tau,p}(z,q)=\sum_{n=0}^{\infty}\frac{(\tau)_{pn}}{\Gamma_q(\sigma n+\delta)}\frac{z^n}{n!}.$$

When $q\to1^-$, the above function is generalized Mittag-Leffler function, which is given by Shukla and Prajapati [5]. Now, We define the function

$$\mathscr{R}_{\sigma,\delta}^{\tau,p}(z,q)=z\Gamma_q(\delta)\,\mathscr{H}_{\sigma,\delta}^{\tau,p}(z,q) \tag{3}$$

and further we define the differential operator $\mathscr{D}_\lambda^m(q,\sigma,\delta,\tau,p):\mathscr{A}\to\mathscr{A}$ by

$$\begin{aligned}
&\mathscr{D}_\lambda^0(q,\sigma,\delta,\tau,p)f(z)=f(z)*\mathscr{R}_{\sigma,\delta}^{\tau,p}(z,q)\\
&\mathscr{D}_\lambda^1(q,\sigma,\delta,\tau,p)f(z)=(1-\lambda)\left(f(z)*\mathscr{R}_{\sigma,\delta}^{\tau,p}(z,q)\right)+\lambda z\left(f(z)*\mathscr{R}_{\sigma,\delta}^{\tau,p}(z,q)\right)'\\
&\qquad\vdots\\
&\mathscr{D}_\lambda^m(q,\sigma,\delta,\tau,p)f(z)=\mathscr{D}_\lambda^1\left(\mathscr{D}_\lambda^{m-1}(q,\sigma,\delta,\tau,p)f(z)\right).
\end{aligned}$$

As we have considered that $f(z)$ is given by Equation (2) then the operator $\mathscr{D}_\lambda^m$ becomes,

$$\mathscr{D}_\lambda^m(q,\sigma,\delta,\tau,p)f(z)=z-\sum_{n=2}^{\infty}\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)a_nz^n \tag{4}$$

$$\text{where,}\qquad \Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)=\frac{(\tau)_{p(n-1)}\Gamma_q(\delta)}{\Gamma_q[\sigma(n-1)+\delta]}\frac{1}{(n-1)!}[\lambda(n-1)+1]^m.$$

Definition 3. *The function $f(z)$ given by (1) is in the class $\mathscr{SR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ if it satisfies following inequality*

$$\left|\frac{(\mathscr{D}_\lambda^m(q,\sigma,\delta,\tau,p)f(z))'-1}{2\mu[(\mathscr{D}_\lambda^m(q,\sigma,\delta,\tau,p)f(z))'-\alpha]-[(\mathscr{D}_\lambda^m(q,\sigma,\delta,\tau,p)f(z))'-1]}\right|<\beta \tag{5}$$

where, $\sigma,\delta,\tau,z\in\mathbb{C},\ \Re(\sigma)>0,\ \Re(\delta)>0,\ \Re(\tau)>0,\ p\in(0,1)\cup\mathbb{N},\ 0<q<1,\ 0\le\lambda,\beta,\mu\le1,\ 0\le\alpha<1.$

Here, we define $\mathscr{TR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)=\mathscr{SR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)\cap\mathbb{T}$.
This present article is followed by techniques used in [8, 9, 10, 11, 13, 14, 15, 16].
In this present paper, we obtain coefficient inequality, extremal properties, radius of starlikeness, radius of convexity, radius of close to convexity and closure theorems of class $\mathscr{TR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ by using above differential operator.

Remark 1. *For $\tau=1$ and $p=1$ we obtain the subclasses studied by [14].*

MAIN RESULTS

Theorem 1. *A function $f(z)$ given by (1) belong to the class $\mathscr{SR}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ if*

$$\sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]|a_n| \leq 2\beta\mu(1-\alpha) \tag{6}$$

where, $\sigma,\delta,\tau,z\in\mathbb{C}$, $\Re(\sigma)>0$, $\Re(\delta)>0$, $\Re(\tau)>0$, $p\in(0,1)\cup\mathbb{N}$, $0<q<1$, $0\leq\lambda,\beta,\mu\leq 1$, $0\leq\alpha<1$.

Proof. Suppose that the inequality holds and consider $|z|=1$, from (5) we have

$$\begin{aligned}
&|(D_{\lambda}^{m}(q,\sigma,\delta,\tau,p)f(z)'-1)| - \beta\left|2\mu\left[(D_{\lambda}^{m}(q,\sigma,\delta,\tau,p)f(z)'-\alpha\right]\right.\\
&\left.-\left[(D_{\lambda}^{m}(q,\sigma,\delta,\tau,p)f(z)'-1\right]\right|\\
&=\left|-\sum_{n=2}^{\infty}\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)na_nz^{n-1}\right|-\beta\left|2\mu\left[1-\sum_{n=2}^{\infty}\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)na_nz^{n-1}-\alpha\right]\right.\\
&\left.+\sum_{n=2}^{\infty}\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)na_nz^{n-1}\right|\\
&\leq\sum_{n=2}^{\infty}\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)n|a_n|-\beta\left[2\mu(1-\alpha)-(2\mu-1)\sum_{n=2}^{\infty}\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)n|a_n|\right]\\
&\leq\sum_{n=2}^{\infty}[1+\beta(2\mu-1)]\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)n|a_n|-2\beta\mu(1-\alpha)\leq 0.
\end{aligned}$$

□

Theorem 2. *A function $f(z)$ given by (2) belong to the class $\mathscr{TR}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ if and only if*

$$\sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]a_n \leq 2\beta\mu(1-\alpha) \tag{7}$$

where, $\sigma,\delta,\tau,z\in\mathbb{C}$, $\Re(\sigma)>0$, $\Re(\delta)>0$, $\Re(\tau)>0$, $p\in(0,1)\cup\mathbb{N}$, $0<q<1$, $0\leq\lambda,\beta,\mu\leq 1$, $0\leq\alpha<1$.
This result is sharp.

Proof. From Theorem 1 it is observed that, we need only to prove the necessity.
Suppose that, $f(z)\in\mathscr{TR}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ then

$$\begin{aligned}
&\left|\frac{(D_{\lambda}^{m}(q,\sigma,\delta,\tau,p)f(z)'-1}{2\mu[(D_{\lambda}^{m}(q,\sigma,\delta,\tau,p)f(z)'-\alpha]-[(D_{\lambda}^{m}(q,\sigma,\delta,\tau,p)f(z)]'-1]}\right|\\
&=\left|\frac{-\sum_{n=2}^{\infty}\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)na_nz^{n-1}}{2\mu(1-\alpha)-(2\mu-1)\sum_{n=2}^{\infty}\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)na_nz^{n-1}}\right|<\beta.
\end{aligned}$$

Since, $\Re(z)\leq|z|$

$$\Re\left\{\frac{\sum_{n=2}^{\infty}\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)na_nz^{n-1}}{2\mu(1-\alpha)-(2\mu-1)\sum_{n=2}^{\infty}\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)na_nz^{n-1}}\right\}<\beta. \tag{8}$$

Choose the values of z on the real axis so that $[(D_{\lambda}^{m}(q,\sigma,\delta,\tau,p)f(z)]'$ is real. Upon clearing denominator in (8) and letting $z\to 1$ through real values, we have

$$\sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]a_n \leq 2\beta\mu(1-\alpha).$$

Hence the proof. □

Corollary 1. *If* $f(z) \in \mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ *if and only if*

$$a_n \leq \frac{2\beta\mu(1-\alpha)}{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]} \tag{9}$$

and equality holds for the function

$$f(z) = z - \frac{2\beta\mu(1-\alpha)}{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]} z^n. \tag{10}$$

This result is sharp.

Theorem 3. *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{2\beta\mu(1-\alpha)}{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]} z^n$$

then $f(z) \in \mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ *if and only if it can be expressed as* $f(z) = \sum\limits_{n=1}^{\infty} \nu_n f_n(z),\ \nu_n \geq 0,\ \sum\limits_{n=1}^{\infty} \nu_n = 1.$

Proof. Suppose,

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \nu_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \nu_n \frac{2\beta\mu(1-\alpha)}{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]} z^n. \end{aligned}$$

Now using Theorem 1, we have

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]}{2\beta\mu(1-\alpha)} \nu_n \frac{2\beta\mu(1-\alpha)}{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]} \\ &= \sum_{n=2}^{\infty} \nu_n = 1 - \nu_n \leq 1. \end{aligned}$$

Thus, $f(z) \in \mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$.
Conversely, let $f(z) \in \mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$.
Using (9) and by setting,

$$\nu_n = \sum_{n=2}^{\infty} \frac{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]}{2\beta\mu(1-\alpha)} a_n, \qquad n \geq 2$$

and $\nu_1 = 1 - \sum\limits_{n=2}^{\infty} \nu_n$.
We have, $f(z) = \sum\limits_{n=1}^{\infty} \nu_n f_n(z)$. □

Theorem 4. *The class* $\mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ *is a convex set.*

Proof. Let

$$f_k(z) = z - \sum_{n=2}^{\infty} a_{nk} z^n, \quad a_{nk} \geq 0,\ k = 1,2$$

belongs to the class $\mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$.
Here, it is sufficient to show that, the function

$$t(z) = \omega f_1(z) + (1-\omega) f_2(z), \qquad 0 \leq \omega < 1$$

belongs to the class $\mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$.
Using Theorem 2, we deduce that

$$\begin{aligned}&\sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]\omega a_{n1}\\ &\qquad\qquad\qquad + \sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)](1-\omega)a_{n2}\\ &\leq \omega 2\beta\mu(1-\alpha)+(1-\omega)2\beta\mu(1-\alpha)\\ &\leq 2\beta\mu(1-\alpha).\end{aligned}$$

Thus, $f(z)\in\mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$.
Hence, the class $\mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ is a convex set. □

Theorem 5. *Let the function $f(z)$ given by (2) belongs to the class $\mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ then $f(z)$ is close to convex of order ρ in the disc $|z|<R_1$,*
where

$$R_1=\inf_{n\geq 2}\left\{\frac{(1-\rho)\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]}{2\beta\mu(1-\alpha)}\right\}^{\frac{1}{n-1}}. \tag{11}$$

This result is sharp with extremal function $f(z)$ given by (10).

Proof. Suppose that $f(z)\in\mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ and f is close to convex function of order ρ, then we have

$$|f'(z)-1|\leq 1-\rho. \tag{12}$$

Indeed we have

$$|f'(z)-1|\leq\sum_{n=2}^{\infty} na_n|z|^{n-1}.$$

Thus,

$$|f'(z)-1|\leq 1-\rho \quad \text{if} \quad \sum_{n=2}^{\infty}\left(\frac{n}{1-\rho}\right)a_n|z|^{n-1}\leq 1. \tag{13}$$

From Theorem 2, we obtain

$$\sum_{n=2}^{\infty}\frac{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]a_n}{2\beta\mu(1-\alpha)}\leq 1. \tag{14}$$

Equation (12) will be true, if

$$\sum_{n=2}^{\infty}\left(\frac{n}{1-\rho}\right)a_n|z|^{n-1}\leq\sum_{n=2}^{\infty}\frac{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]a_n}{2\beta\mu(1-\alpha)}.$$

Solving above Inequality for $|z|$, we obtain

$$|z|\leq\left\{\frac{(1-\rho)\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]}{2\beta\mu(1-\alpha)}\right\}^{\frac{1}{n-1}}. \tag{15}$$

Thus, Theorem 5 follows from (15). □

Theorem 6. *Let the function $f(z)$ given by (2) belongs to the class $\mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ then $f(z)$ is starlike of order ρ in the disc $|z|<R_2$,*
where

$$R_2=\inf_{n\geq 2}\left\{\frac{(1-\rho)n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]}{(n-\rho)2\beta\mu(1-\alpha)}\right\}^{\frac{1}{n-1}}. \tag{16}$$

This result is sharp with extremal function $f(z)$ given by (10).

Proof. Suppose, $f(z) \in \mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ and f is starlike of order ρ, then we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho. \tag{17}$$

Indeed we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}.$$

Thus,

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\rho \quad \text{if} \quad \sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)a_n|z|^{n-1} \leq 1. \tag{18}$$

Now, using the fact that $f(z) \in \mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ if and only if

$$\sum_{n=2}^{\infty}\frac{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]a_n}{2\beta\mu(1-\alpha)} \leq 1. \tag{19}$$

Equation (17) will be true if

$$\sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)a_n|z|^{n-1} \leq \sum_{n=2}^{\infty}\frac{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]a_n}{2\beta\mu(1-\alpha)}.$$

i.e.

$$|z| \leq \left\{\frac{(1-\rho)n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]}{(n-\rho)2\beta\mu(1-\alpha)}\right\}^{\frac{1}{n-1}}. \tag{20}$$

Thus, Theorem 6 follows from (20). □

Theorem 7. *Let the function $f(z)$ given by (2) belongs to the class $\mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ then $f(z)$ is convex of order ρ in the disc $|z| < R_3$,*
where

$$R_3 = \inf_{n\geq 2}\left\{\frac{(1-\rho)\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]}{(n-\rho)2\beta\mu(1-\alpha)}\right\}^{\frac{1}{n-1}}. \tag{21}$$

This result is sharp with extremal function $f(z)$ given by (10).

Proof. Following same technique of Theorem 6 and using the fact that function $f(z)$ is convex if and only if $zf'(z)$ is starlike function. □

Theorem 8. *If*

$$t(z) = z - \sum_{n=2}^{\infty}t_n z^n \tag{22}$$

and

$$r(z) = z - \sum_{n=2}^{\infty}r_n z^n \tag{23}$$

are in the class $\mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ then $u(z) = z - \frac{1}{2}(t_n + r_n)z^n$ is also in the class $\mathscr{T}\mathscr{R}_{\lambda}^{m}(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$.

Proof. Let $t(z), r(z) \in \mathscr{TR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$.
Using Theorem 2, we have

$$\sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]t_n \leq 2\beta\mu(1-\alpha) \tag{24}$$

$$\sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]r_n \leq 2\beta\mu(1-\alpha). \tag{25}$$

From Equation (24) and (25), we deduce that

$$\frac{1}{2}\sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)](t_n+r_n) \leq 2\beta\mu(1-\alpha).$$

Thus, $u(z) = z - \frac{1}{2}(t_n+r_n)z^n$ is also in the class $\mathscr{TR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$. □

Theorem 9. *For $t(z), r(z) \in \mathscr{TR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$, the weighted mean*

$$\omega_d(z) = \frac{(1-d)t(z)+(1+d)r(z)}{2}, \qquad d \in \mathbb{N}, z \in \mathscr{U} \tag{26}$$

is in $\mathscr{TR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$.

Proof. Let $t(z)$ and $r(z)$ given by Eqaution (22) and (23) belongs to the class $\mathscr{TR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$.
Now, we define the weighted mean ω_d of functions $t(z)$ and $r(z)$ as

$$\omega_d(z) = z - \sum_{n=2}^{\infty} \frac{(1-d)t_n+(1+d)r_n}{2} z^n.$$

To show that $\omega_d \in \mathscr{TR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$, it is sufficient to show that

$$\sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]\left[\frac{(1-d)t_n+(1+d)r_n}{2}\right] \leq 2\beta\mu(1-\alpha).$$

Consider the left hand side term and using Theorem 2, we deduce that

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]\left[\frac{(1-d)t_n+(1+d)r_n}{2}\right] \\
&= \sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]\frac{(1-d)}{2}t_n \\
&\qquad\qquad + \sum_{n=2}^{\infty} n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]\frac{(1-d)}{2}r_n \\
&\leq \frac{(1-d)}{2}2\beta\mu(1-\alpha) + \frac{(1-d)}{2}2\beta\mu(1-\alpha) \\
&\leq 2\beta\mu(1-\alpha).
\end{aligned}$$

Thus, $\omega_d(z) \in \mathscr{TR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$. □

Theorem 10. *The function $t(z)$, $r(z) \in \mathscr{TR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$ then the convolution $Y(z) = t(z) * r(z) \in \mathscr{TR}_\lambda^m$ $(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$*
if $|r_n| \leq 1$.

Proof. Assume that $t(z)$ and $r(z)$ given by Eqaution (22) and (23) belongs to the class $\mathscr{TR}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$.
Then, the convolution of $t(z)$ and $r(z)$ is given by

$$Y(z) = t(z) * r(z) = z - \sum_{n=2}^{\infty} t_n r_n z^n.$$

If $r_n \leq 1$ then $t_n r_n \leq t_n \ \forall n$.
Let us consider the term and using Corollary 1, we get

$$\sum_{n=2}^{\infty} \frac{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]}{2\beta\mu(1-\alpha)} t_n r_n \leq \sum_{n=2}^{\infty} \frac{n\Psi_n^m(\lambda,q,\sigma,\delta,\tau,p)[1+\beta(2\mu-1)]}{2\beta\mu(1-\alpha)} t_n \leq 1.$$

Hence, $Y(z) \in \mathscr{T}\mathscr{R}_\lambda^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$. □

CONCLUSION

Recent years have seen a significant increase in interest in q-calculus because of it's numerous applications in domains like as mathematical and quantum physics. We defined a new differential operator associated with the generalized q-Mittag-Leffler function in the current study. We introduced a new subclass of analytic and univalent functions $\mathscr{T}\mathscr{R}_\rho^m(q,\sigma,\delta,\tau,p,\alpha,\beta,\mu)$. Using this operator coefficient theorems, extremal properties, radius of starlikeness, radius of convexity, radius of close-to-convexity, closure theorems for the newly formed class were among the properties that were examined.

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REFERENCES

1. H. Silverman , Univalent functions with negative coefficients, "Proceedings of the American mathematical society", 51(1), pp.109-116, (1975).
2. G. M. Mittag-Leffler, Sur la nouvelle,"Comptes Rendus deIâAcademie des Sci. Paris Series", 11(137), pp.537-539, (1903).
3. A. Wiman, "Uber den fundamentalsatz in der teorie der funktionen", *ActaMath.*, 29, pp.191-201, (1905).
4. T. R. Prabhakar, "A singular integral equation with a generalized Mittag-Leffler function in the kernel", *Yokohamamath.J*, 19(1), pp.7-15, (1971).
5. A. K. Shukla and J. C. Prajapati, "On a generalization of Mittag-Leffler function and its properties", *Journal of mathematical analysis and applications*, 336(2), pp.797-811, (2007).
6. F. Jackson, "On q-functions and a certain difference operator", *trans. Roy. Soc. Edinburgh.*, 46, pp.253-281, (1908).
7. S. H. Hadi, M. Darus and J. R. Lee, "Some geometric properties of multivalent functions associated with a new generalized q-Mittag-Leffler function", *AIMS Mathematics*, 7(7), pp.11772-11783, (2022).
8. M. K. Aouf and N. E. Cho, " On a certain subclass of analytic functions with negative cefficients", *Turkish Journal of Mathematics*, 22(1), pp.15-32, (1998).
9. M. K. Aouf, H. M. Hossen and A. Y. Lashin, "A class of univalent functions defined by using Hadamard product", *Matematički vesnik*, 55(3-4), pp.83-96, (2003).
10. M. K .Aouf, A. K. Mostafa, A. Y. Lashin, and B. M. Munassar, "On a new class of univalent functions with negative coefficients defined by Dziok-Srivastava Operator", *J. Math. Anal. Appl*, 2, pp.117-131, (2014).
11. S. Owa and M. K. Aouf, "On subclasses of univalent functions with negative coefficients", *East Asian mathematical journal*, 4, pp.57-73, (1988).
12. G. S. SăLăgean, "Integral properties of certain classes of analytic functions with negative coefficients", *International journal of mathematics and mathematical sciences* , 2005, pp.125-131, (2005).
13. M. K. Aouf and H. M. Srivastava, "Certain families of analytic functions with negative coefficients", *dspace.library.uvic.ca*, (1994).
14. S. B. Joshi, S. S. Nalavade and S. S. Joshi, "Certain Class Of Univalent Functions Associated With Mittag-Leffler Function", *Journal of Indian Acad. Math.*, 44(2), pp.145-155, (2022).
15. O. Altintas and S. Owa, "On subclasses of univalent functions with negative coefficients", *East Asian mathematical journal*, 4, pp.41-56, (1988).
16. A. O. Lasode, T. O. Opoola, I. Al-Shbeil, T. G. Shaba, and H. Alsaud, "Concerning a Novel Integral Operator and a Specific Category of Starlike Functions", *Mathematics* , 11(21), pp.4519-4519, (2023).